Coevolution of agents and networks: Opinion spreading and community disconnection

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Abstract

We study a stochastic model for the coevolution of a process of opinion formation in a population of agents and the network which underlies their interaction. Interaction links can break when agents fail to reach an opinion agreement. The structure of the network and the distribution of opinions over the population evolve towards a state where the population is divided into disconnected communities whose agents share the same opinion. The statistical properties of this final state vary considerably as the model parameters are changed. Community sizes and their internal connectivity are the quantities used to characterize such variations.

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In a complex system, the spontaneous emergence of collective non-equilibrium behaviour, such as coherent spatiotemporal structures or synchronized dynamics, is driven by mechanisms which involve both the interactions between the system's components and external influxes [1,2]. In segregation phenomena, a form of self-organization well known to physicists and chemists, an ensemble of interacting elements becomes split into subensembles whose components share certain individual states. Segregation occurs also in biological and social systems [3,4,5], where it plays a crucial role in sustaining diversity at many levels –cellular, functional, organizational, ecological, cultural. Though it is usually associated with phase separation in space, segregation not always takes place in the spatial domain. In human societies, for instance, two or more subpopulations or communities may coexist in the same geographical region and, yet, exhibit mutually excluding cultural traits [6]. With respect to those traits –which may involve religious beliefs, professional or generational jargons, artistic inclinations, etc.– such communities are scarcely interacting,

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and can be effectively considered as segregated from each other. A key mechanism in this kind of social segregation is the feedback between the construction of agreement within a community and the enhancement of distinctions with other communities: specific traits become better established as differences between communities develop and grow.

In the last years, physicists have been increasingly interested in the dynamical and statistical modelling of complex systems of biological and sociological inspiration as populations of agents whose interaction patterns are described through graphs, or networks. Much attention has been paid to dynamical processes defined over quenched networks [2,7], and to network growth –both purely stochastic [8] or driven by adaptive mechanisms [9]– with emphasis in the statistical properties of the resulting patterns. On the other hand, the possible coevolution of the network structure and the dynamics taking place over them seems to have been less studied (see, however, Ref. [10]). This kind of coevolution is at the basis of the feedback phenomenon pointed out above, where the formation of internally homogeneous subpopulations and the weakening of their mutual interactions enhance each other.

The aim of this letter is to present a very simple model for coevolution of a population of agents and their interaction network. The agents' dynamics is based on an elementary model of opinion spreading [11]. The population starts in a situation where every agent is able to interact with any other agent, and evolves towards a segregated state with disconnected communities. Interactions between agents with similar or different opinions are respectively favoured or penalized. In spite of the simplicity of the evolution rules, the population can reach a variety of social patterns, which range from splitting into two communities of similar sizes and opposite opinions, to a single large community with homogeneous opinion. Typical connectivities and opinion distributions, resulting from the combined evolution of the network and the agent states, change considerably with the control parameters.

The system consists of N agents situated at the nodes of a network. Initially, the network is fully connected, so that any pair of agents can potentially interact. The individual state of each agent is assigned at random one of two possible values, say $+1$ or -1, with equal probability. Individual states represent the agents' opinions, which may eventually change by interaction between agents connected by the network.

The evolution runs as follows. At each step, a pair of connected agents is chosen at random from the whole population. If both agents have the same opinion, nothing happens. Otherwise, with probability p_1 either agent adopts the other agent's opinion, so that the two opinions become identical. With the complementary probability $1 - p_1$, opinions are not changed. In this case, however, the link between both agents is broken with probability p_2, and the

interaction network looses one of its edges. These rules are successively applied until no further changes are possible. Since network edges are irreversibly deleted by the evolution, a frozen final state is eventually reached where, generally, the network is split into disconnected subsets. Within each of these communities all agents share the same opinion.

The frequencies of the individual events that drive the dynamics depend on the probabilities p_1 and p_2. With respect to the evolution of the system, these probabilities are independent control parameters. The statistical properties of the final state, however, are completely determined by the relative frequencies of the different processes that effectively change the state of the system. In other words, they depend on p_1 and p_2 through certain algebraic combinations only. To realize this, let $p_-(t)$ be the fraction of network links connecting agents with different opinions. The probability that any agent changes its opinion at a given step is $\pi_1(t) = p_-(t)p_1$, and the probability that a connection is broken is $\pi_2(t) = p_-(t)(1 - p_1)p_2$. The sum $\pi(t) = \pi_1(t) + \pi_2(t)$ is the probability per evolution step that any change takes place, and thus fixes an overall evolution time scale. If p_1 and p_2 vary in such a way that the ratios $q = \pi_1(t)/\pi(t)$ and $1 - q = \pi_2(t)/\pi(t)$ are kept constant, such overall time scale will change, but the relative frequency of the two processes will be the same, giving rise to statistically equivalent final states. Thus, the only independent combination of the probabilities p_1 and p_2 relevant to the determination of the final state is

$$q = \frac{\pi_1(t)}{\pi(t)} = \frac{p_1}{p_1 + (1 - p_1)p_2}. \tag{1}$$

The behaviour at the two extreme values of q is immediately assessed. For $q = 0$ (i.e. $p_1 = 0$), where no opinion changes take place, the final structure consists of two mutually disconnected communities with similar sizes and opposite opinions. Internally, each of them stays fully connected, so that the total number of remaining connected pairs R is close to $2(N/2)(N/2-1)/2 \approx N^2/4$. For $q = 1$ (i.e. $p_1 = 1$), on the other hand, interacting agents with initially opposite opinions always reach consensus, so that no interaction links are broken. At the final state, all agents share the same opinion and the network is still fully connected, with its $N(N - 1)/2 \approx N^2/2$ links intact.

These two limits suggest that the fraction of remaining connected pairs, $r = 2R/N(N - 1)$, provides a first quantitative characterization of the final state. Figure 1 shows r as a function of q, for systems of various sizes N. For each value of q, the fraction r has been obtained as the average over 500 to 5000 realizations of the initial condition and the evolution. Rather unexpectedly, we find that r reaches a minimum for an intermediate value of q. The position q_{min} and depth r_{min} of this minimum depend on N, and seem to tend to zero as the population grows, as shown by the plot in insert (b). The network connectivity
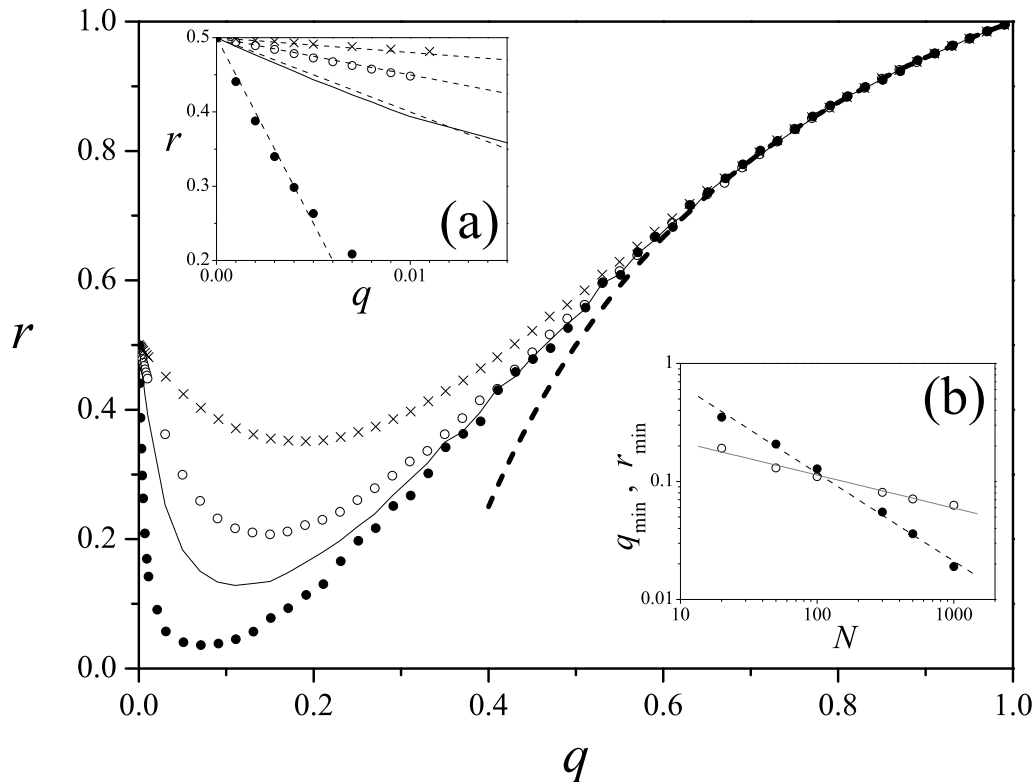
Fig. 1. The fraction r of remaining links as a function of the parameter q, for $N = 20$ (×), 50 (○), 100 (full line, and 500 (●). The dashed line represent the analytical approximation for large q discussed in the text. Insert (a): Close-up of the same data for small q. The dashed lines represent the small-q approximation discussed in the text. Insert (b): The position $q_{\min}$ (○) and the depth $r_{\min}$ (●) of the minimum of r as functions of N. Straight lines are least-square fittings with slopes -0.28 (full line) and -0.75 (dashed line).

at the minimum is considerably lower than at the extreme values $q = 0$ and 1. For $N = 100$, for instance, we have $r_{\min} \approx 0.13$, which implies an average connectivity of about 13 links per site. In this intermediate regime, thus, the network is poorly connected and the population structure correspondingly degraded.

The minimum at $q_{\min}$ defines two regimes in the parameter q. To the left ($q < q_{\min}$), as expected from the behaviour for $q = 0$, the population becomes split into two separate communities with similar sizes and opposite opinions. As q grows towards $q_{\min}$, the internal connectivity of these communities decreases. Simultaneously, several small separate clusters, each of them containing just a few agents (typically, less than 5 for $N = 500$), are also found at the final state.

To the right of the minimum, we still have realizations where the population splits into disconnected communities. The two largest communities always
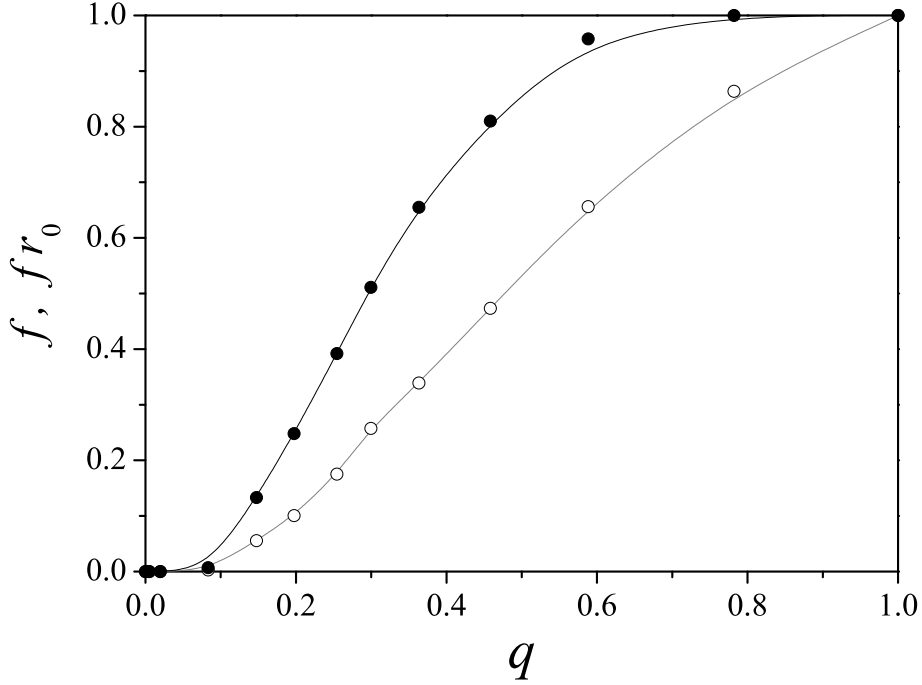
Fig. 2. The fraction f of realizations ending in a single community ($\bullet$), and the contribution fr_0 of those realizations to the fraction of remaining links ($\circ$), where r_0 is the average number of remaining links in a single-community final state. Curves ar Bézier splines, added for clarity.

have opposite opinions, but their size is much more variable than for $q < q_{\min}$. Moreover, the number and size of small communities grow. At the same time, it becomes increasingly frequent to find realizations where the whole population stays connected into a single community. In these realizations, the final network is not fully connected, but its connectivity is significantly larger than in the cases where separate communities are formed. As a matter of fact, it is the contribution of these realizations which determines the growth of the fraction of connected pairs r for $q > q_{\min}$. As q keeps growing, the fraction of realizations with a single-community final state increases and, eventually, all realizations end on such states, with a homogeneous opinion all over the system. Figure 2 shows the fraction f of realizations where the population reaches a single-community final state as a function of q for $N = 500$, and the contribution of those realizations to the fraction of remaining links.

In order to compare the regimes at both sides of the minimum, we have chosen to study in more detail the final structure of a population of size $N = 500$ for two values of q where the fraction of remaining connected pairs r attains similar levels, namely, $q = 5 \times 10^{-3}$ and 0.3 ($r \approx 0.27$; see Fig. 1). Figure 3

shows, for the two chosen values of q, the size of the second largest community (N_2) as a function of the size of the largest community (N_1), for 10^3 realizations. Generally, $N_2 \neq 0$, except in realizations where the final state consists of a single community, where $N_1 = N$ and N_2 vanishes. The straight full line stands for the function $N_2 = N - N_1$, so that the vertical distance from each dot to the line represents the size $N_0 = N - N_1 - N_2$ of the population not included in the two largest communities for the corresponding realization. The dashed line, in turn, corresponds to $N_2 = (N - N_1)/2$. Dots below this line represent realizations where the second largest community is smaller than N_0.

For $q < q_{\min}$, we find that the size of the largest community reaches, at most, $N_1 \approx 340 \approx 0.7N$. Moreover, most dots lie over the straight line, which implies that essentially all agents are in one of the two largest communities $(N_0 \approx 0)$. For $q > q_{\min}$, on the other hand, the largest community can reach the maximum size $N_1 = 500 = N$, indicating that some realizations already correspond to single-community final states. Now, all sizes between $N_1 \lesssim N/2$ and N seem to be possible. Except in the close proximity of $N_1 = N$, where the largest community comprises practically the whole population, dots are sensibly below the full line. The population N_0 not belonging to the two largest communities is typically around $0.1N$. Moreover, for most realizations where $N_1 \gtrsim 400 = 0.8N$, this population is above the size of the second largest community. In this situation, we can no longer properly speak of splitting into two main communities. The final structure is in fact closer to the single-community state, with the addition of several small separated communities.

The distribution of internal connectivities in the resulting communities is also strongly dependent on the parameter q. Figure 4 shows the number of links P_i within each community as a function of the community size N_i, for the two largest communities in each realization $(i = 1, 2)$. Two different symbols identify the largest ($\circ$) and the second largest ($\bullet$) communities. A third symbol ($\blacksquare$) is used for realizations with a single-community final state $(N_1 = N)$. The full curve represents the function $P_{1,2} = N_{1,2}(N_{1,2} - 1)/2$, corresponding to fully connected networks. The dashed curve stands for $P_{1,2} = N_{1,2} - 1$, the minimum number of links in a connected network.

For $q < q_{\min}$, the number of remaining links in each of the two largest communities exhibits a limited dispersion between realizations –just like their size $N_{1,2}$. As a result, all the realizations are represented by dots in a rather compact cloud. Within each community, roughly one half of the total possible connections are actually present, $P_{1,2} \approx N_{1,2}(N_{1,2} - 1)/4$, so that the average connectivity per site is about $N_{1,2}/2$. In this situation, thus, the internal connectivity of both communities is quite high.

For $q > q_{\min}$, in contrast, the number of remaining links is broadly distributed. For practically all sizes, the number of links of second-largest communities
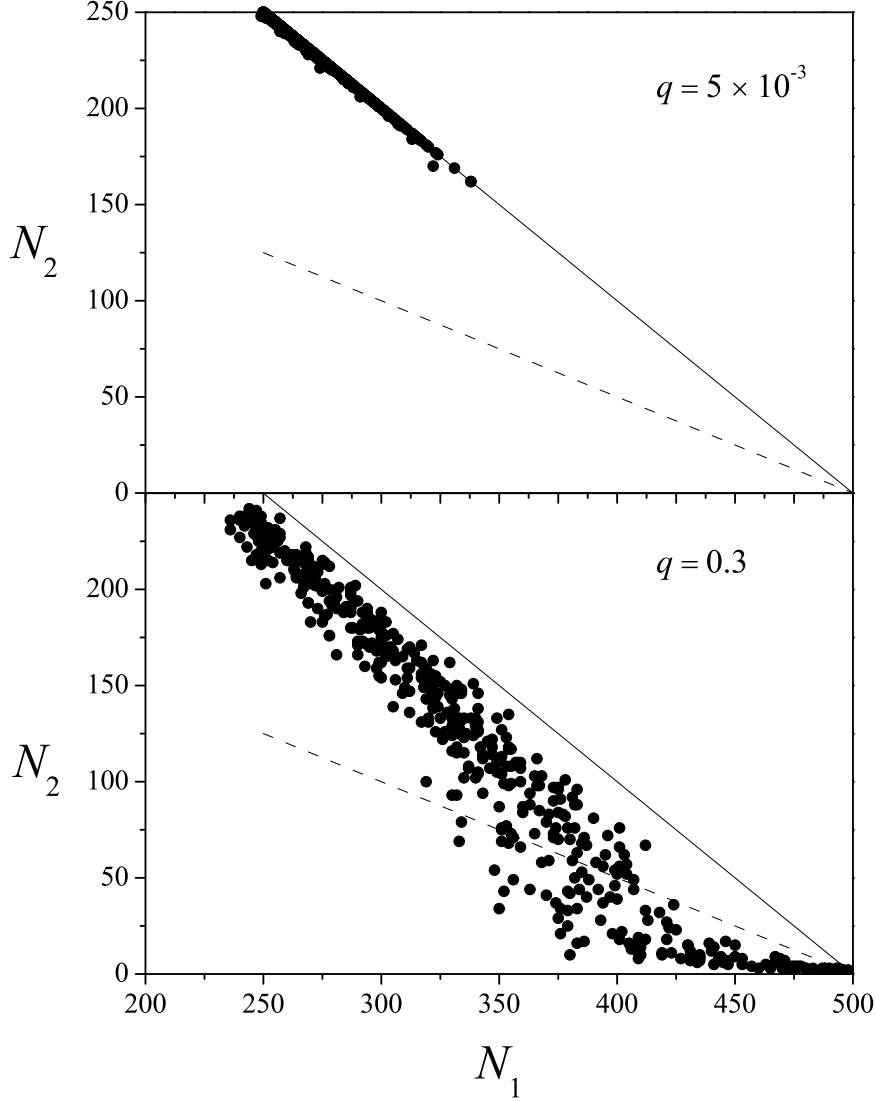
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Fig. 3. The size N_2 of the second largest community as a function of the size N_1 of the largest community, for two values of the control parameter q, in a series of 10^3 realizations of a system with $N = 500$ agents. Each dot corresponds to a single realization. The straight and the dashed lines represent, respectively, the functions $N_2 = N - N_1$ and $N_2 = (N - N_1)/2$.

is just above the minimum, $P_{1,2} = N_{1,2} - 1$, indicating a barely connected structure. The average connectivity per site within these communities is, in fact, close to one. The number of links in the largest communities, on the other hand, are always considerably above the minimum. However, even the
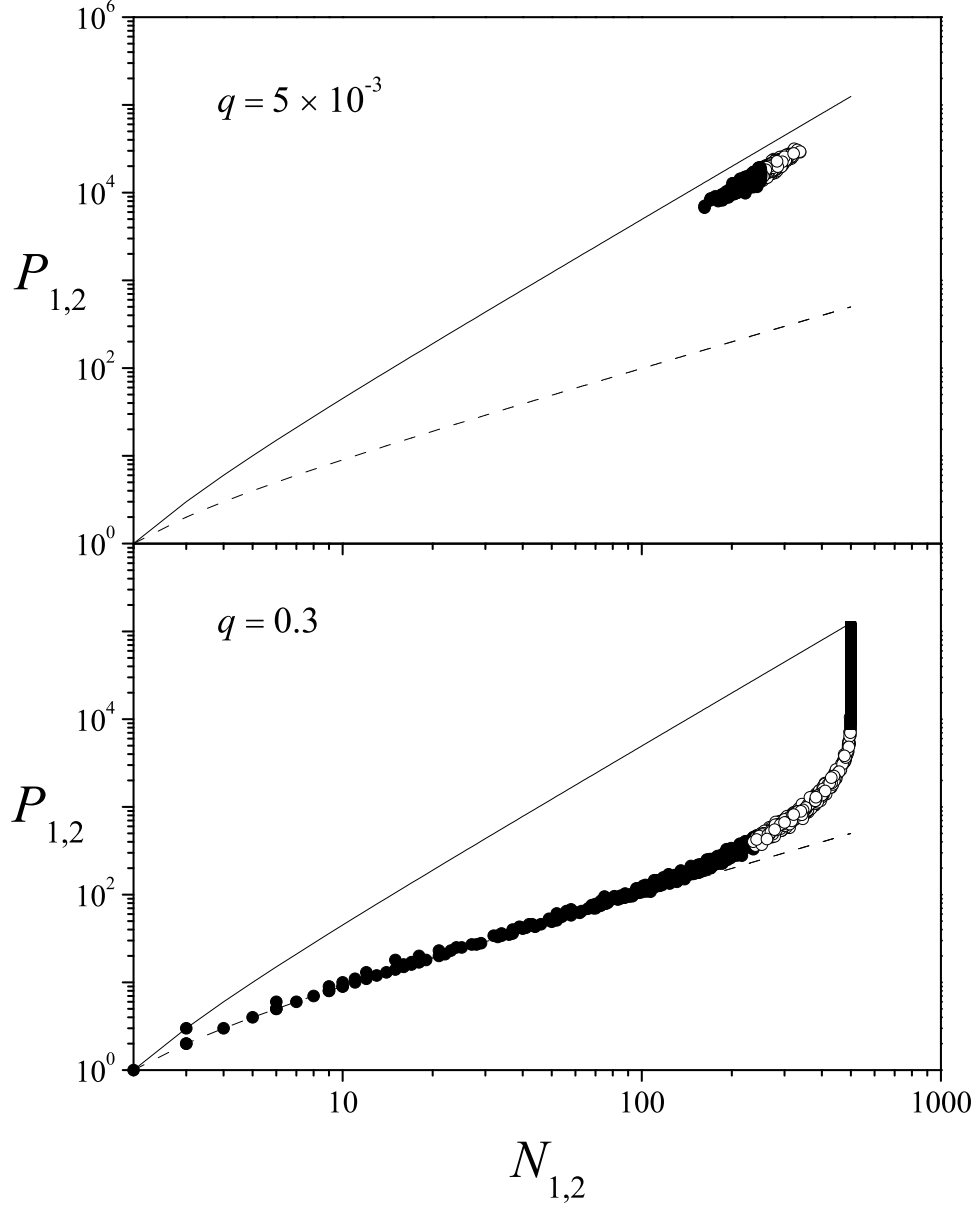
Fig. 4. The number of remaining links P_i within the largest ($\circ$) and the second largest ($\bullet$) community ($i = 1, 2$, respectively), as a function of the community size N_i, for two values of q, in a series of 10^3 realizations of a system of $N = 500$ agents. Realization with single-community final states are indicated by a different symbol ($\blacksquare$). The full curve in each plot represents the function $P_{1,2} = N_{1,2}(N_{1,2} - 1)/2$; the dashed curve is $P_{1,2} = N_{1,2} - 1$.

better connected among these communities are still far from the situation of full connectivity. For realizations where $N_1 \lesssim N/2$, P_1 reaches at most 10% of its maximum value. The number of links almost attains the maximum in some of the realizations where the population stays connected in a single community. These realizations, in fact, fill the gap between the best connected communities in the case of split populations and fully connected networks. Interestingly, while the above description emphasizes the differences between the internal structure of second-largest communities, largest communities, and single-community populations, the fact that all the dots in the lower panel of Fig. 4 seem to lie over a smooth curve suggest that those different structures belong to a sole class. The internal connectivity varies continuously as the community size changes.

Obviously, the coevolution of the distribution of opinions and the interaction network creates strong correlations between the individual states of those agent pairs that remain connected. These correlations make an accurate analytical treatment of our model particularly difficult. Some of the features found in the numerical results, however, admit an analytical explanation under suitable hypotheses. Let us begin by the regime of large q where, as we have seen, most realizations end in a single-community state. The evolution towards a homogeneous opinion all over the population is fully driven by fluctuations. In fact, the number N_+ of agents with opinion $+1$, for instance, performs a random walk, $N_+ \rightarrow N_+ \pm 1$, each step an opinion changes. Starting at $N_+ \approx N/2$, the process ends when N_+ reaches either 0 or N. From standard results on first-passage time distributions in random walks [12], we find that the average number of active steps needed to reach one of these extreme values is $S = N^2/4p_1$. An active step is here defined as an evolution step where any change, either the flip of an individual opinion or the deletion of a network link, occurs. In an active step, a link is deleted with probability $(1 - p_1)p_2$. If we assume that this latter event is not correlated with opinion changes, the total number of links deleted in S active steps is $N^2(1 - p_1)p_2/4p_1$. Taking into account that the initial number of links is, approximately, $N^2/2$, we have

$$ r \approx \frac{1}{N^2/2} \left[\frac{N^2}{2} - \frac{N^2}{4} \frac{(1 - p_1)p_2}{p_1} \right] = 1 - \frac{1-q}{2q}. \tag{2} $$

Note that this estimation for the fraction of remaining links is independent of N. It is represented by the dashed curve in Fig. 1. We see that the agreement with numerical results is excellent for large q. While neglecting correlations between link deletion and opinion changes is not strictly correct –as a matter of fact, links are deleted precisely in those steps where opinions *do not* change– for $q \approx 1$ deletion events are so rare that such correlations can hardly accumulate into a relevant effect before the final state is reached.

For small q, on the other hand, the final state always corresponds to essentially

two large communities with opposite opinions. Since, in this limit, p_1 is small, most links connecting agents with different opinions are deleted before any significant number of opinion flips occurs. At any time, thus, essentially all links connect agents with the same opinion, except just after an opinion flip: when an agent's opinion changes, most of its links are now connections with agents with the opposite opinion. The total number P_+ of links connecting agents with the same opinion remains constant until an opinion flip takes place. In such event, the decrease in P_+ is given by the number of links of the agent whose opinion is changing, which can considerably vary among agents. As a kind of mean-field approach, we assume that this number is proportional to the average per-agent number of links connecting agents with the same opinion, $2P_+/N$, so that $\dot{P}_+ = -2\alpha p_1 P_+/N$. The derivative is performed with respect to a time variable whose units are active steps, and the heuristic factor α represents the proportionality assumed above. The solution to this equation is

$$P_+(s) = \frac{N^2}{4} \exp\left(-\frac{2\alpha p_1}{N}s\right),\tag{3}$$

where s is the number of elapsed active steps. Moreover, as discussed above, the total number of links after s active steps is

$$P(s) = \frac{N^2}{2} - (1 - p_1)p_2 s.\tag{4}$$

The evolution ends when $P_+(s) = P(s)$, so that the fraction r of remaining links is determined by the equation

$$r = \frac{1}{2}\exp\left[-\alpha N\frac{q}{1-q}(1-r)\right].\tag{5}$$

Insert (a) of Fig. 1 shows fittings of the numerical results for small q, obtained from this equation with $\alpha = 0.4$. The fact that α does not depend on the size N implies that r depends on q and N through their product qN. This explains, at least qualitatively, that the minimum in r shifts to smaller q as N grows. This shift is a direct consequence of the different dependence on N of the two mechanisms which drive the system for small q. While at each active step only one link can be deleted, the number of links connecting agents with equal or different opinions can change in a quantity of order N.

In summary, we have shown that our simple model of coevolution for a population of agents and the underlying interaction network gives rise to an interesting variety of population structures. The evolution rules represent the spreading of a bivaluate opinion on a network whose links can break when

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agents do not succeed at reaching an opinion agreement. The final population structure consists, typically, of a set of separate communities, each of them containing agents with the same opinion. The resulting structures can be divided into three main classes: (i) two internally well connected communities with similar sizes and opposite opinions; (ii) a single community containing all the population; and (iii) a well connected community with, typically, more than half the population, accompanied by a set of poorly connected smaller communities. In this latter class, the two largest communities always have opposite opinions.

While a clear boundary between the three classes cannot be unambiguously drawn, class (i) on the one hand, and (ii) and (iii) on the other, are observed within different ranges of the control parameter $q \in [0, 1]$. The cross-over region between these two ranges moves towards $q = 0$ when the population size N grows, so that the small-q regime would disappear for $N \to \infty$. This effect can be avoided if, when two agents fail to reach an opinion agreement, not only their mutual connection but a given fraction of their links to other agents is also broken. In this way, deletion of links and opinion change would have the same dependence on the system size. Preliminary numerical results of this extension of the model have already been obtained, and will be published in a forthcoming paper. Another extension, in the direction of making the model more realistic, would allow for the creation of links between disconnected agents. This process would drive the population to a dynamic asymptotic state, independent of the initial structure, where communities could form, aggregate, exchange agents, and disappear, as known to happen in real social systems.

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